A New Cyclic Module for Hopf Algebras [*]

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Abstract

We define a new cyclic module, dual to the Connes-Moscovici co-cyclic module, for Hopf algebras, and give a characteristic map for coactions of Hopf algebras. We also compute the resulting cyclic homology for cocommutative Hopf algebras, and some quantum groups.

Keywords. Cyclic homology, Hopf algebras.

1 Introduction

In their study of the index theory of transversally elliptic operators [2], Connes and Moscovici developed a cyclic (co-)homology theory for Hopf algebras, which can be considered as an extension of group homology and Lie algebra homology to Hopf algebras and in particular to quantum groups. This theory was further explained and developed along purely algebraic lines in [1, 5]. One of the main tools in [2, 1] is a noncommutative characteristic map $HC^*_{(\delta,\sigma)}(\mathcal{H}) \longrightarrow HC^*(A)$, for a Hopf algebra $\mathcal{H}$ and a $\mathcal{H}$-module algebra A endowed with an invariant trace.

There is however a need for a dual theory to be developed. This is justified for example when one studies coactions of Hopf algebras and also by the fact, first observed by M. Crainic, that for group algebras and in general for Hopf

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[*]The results of this paper were announced in the AMS meeting in Toronto, September 23-24, 2000.

algebras with a normalized Haar integral (i.e. cosemisimple Hopf algebras), that both cyclic homology and cohomology are trivial [5].

In this paper we define a new cyclic module for Hopf algebras. We define the characteristic map for the coaction of Hopf algebras and also prove an analogue of Karoubi's theorem for cocommutative Hopf algebras. We also compute our theory for quantum groups $U_q(sl_2)$ and $A(SL_q(2))$. It would be very interesting to compute this theory for other quantum groups. In [5] one can find a computation of Hopf algebra cyclic cohomology, in the sense of Connes-Moscovici, for the quantum group $U_q(sl_2)$. The method however is very different from ours. We note that a similar cyclic module is also independently considered by R. Taillefer [11], for different reasons.

We would like to thank the referee whose suggestions and comments improved our presentation, specially in the last section where our original formulation of Theorem 5.1 and Corollary 5.2 were inaccurate.

2 Cyclic Module of Hopf Algebras

Let $(\mathcal{H}, \mu, \eta, \delta, \epsilon, S)$ be a Hopf algebra over a commutative ring k, where $\mu, \eta, \delta, \epsilon, S$ denote the product, unit map, coproduct, counit and the antipode of $\mathcal{H}$, respectively. Let σ be a grouplike element of $\mathcal{H}$ and $\delta : \mathcal{H} \longrightarrow k$ be a character for $\mathcal{H}$ as in [1]. The pair (δ, σ) is called a modular pair if $\delta(\sigma) = id$, and a modular pair in involution if

$$\widetilde{S}_\sigma^2 = id, \tag{1}$$

where

$$\widetilde{S}_\sigma(h) = \sigma \sum_{(h)} \delta(h^{(2)}) S(h^{(1)}).$$

We have used Sweedler's notation [10] i.e., $\Delta(h) = \sum_{(h)} h^{(1)} \otimes h^{(2)}$. We will associate a cyclic module to any Hopf algebra $\mathcal{H}$ over k if $\mathcal{H}$ has a modular pair (δ, σ) in involution. This cyclic module somehow can be seen as the dual of the cocyclic module introduced in [1, 2] by A. Connes and H. Moscovici. First, consider S_σ (where $S_\sigma(h) = \sigma S(h)$) which has the properties:

$$S_\sigma(h_1 h_2) = S_\sigma(h_2) S(h_1)$$

$$S_\sigma(1) = \sigma$$

$$\Delta S_\sigma(h) = \sum_{(h)} S_\sigma(h^{(2)}) \otimes S_\sigma(h^{(1)})$$

$$\epsilon(S_\sigma(h)) = \epsilon(h).$$

Using ϵ and δ one can endow k with an $\mathcal{H}$-bimodule structure, i.e.,

$$\delta \otimes id : \mathcal{H} \otimes k \longrightarrow k \quad and \quad id \otimes \epsilon : k \otimes \mathcal{H} \longrightarrow k.$$

Our cyclic module as a simplicial module is exactly the Hochschild complex of the algebra $\mathcal{H}$ with coefficients in k, where k is an $\mathcal{H}$-bimodule as above. So if we denote our cyclic module by $\widetilde{\mathcal{H}}^{(\delta,\sigma)}$, we have $\widetilde{\mathcal{H}}_n^{(\delta,\sigma)} = \mathcal{H}^{\otimes n}$ and $\widetilde{\mathcal{H}}_0^{(\delta,\sigma)} = k$. Its faces and degeneracies are as follows:

$$
\begin{aligned}
\delta_0(h_1 \otimes h_2 \otimes ... \otimes h_n) &= \epsilon(h_1)h_2 \otimes h_3 \otimes ... \otimes h_n \\
\delta_i(h_1 \otimes h_2 \otimes ... \otimes h_n) &= h_1 \otimes h_2 \otimes ... \otimes h_i h_{i+1} \otimes ... \otimes h_n \quad 1 \leq i \leq n-1 \\
\delta_n(h_1 \otimes h_2 \otimes ... \otimes h_n) &= \delta(h_n)h_1 \otimes h_2 \otimes ... \otimes h_{n-1} \\
\sigma_0(h_1 \otimes h_2 \otimes ... \otimes h_n) &= 1 \otimes h_1 \otimes ... \otimes h_n \\
\sigma_i(h_1 \otimes h_2 \otimes ... \otimes h_n) &= h_1 \otimes h_2 ... \otimes h_i \otimes 1 \otimes h_{i+1} ... \otimes h_n \quad 1 \leq i \leq n-1 \\
\sigma_n(h_1 \otimes h_2 \otimes ... \otimes h_n) &= h_1 \otimes h_2 \otimes ... \otimes h_n \otimes 1.
\end{aligned}
$$

To define a cyclic module it remains to introduce an action of the cyclic group on our module. Our candidate is

$$\tau_n(h_1 \otimes h_2 \otimes ... \otimes h_n) = \sum \delta(h_n^{(2)})(S_\sigma(h_1^{(1)} h_2^{(1)} ... h_n^{(1)}) \otimes h_1^{(2)} \otimes ... \otimes h_{n-1}^{(2)}).$$

Theorem 2.1. *Let* $(\mathcal{H}, \mu, \eta, \delta, \epsilon, S)$ *be a Hopf algebra over* k *with a modular pair* (δ, σ) *in involution. Then* $\widetilde{\mathcal{H}}^{(\delta,\sigma)}$ *with operators given above defines a cyclic module. Conversely, if* $\delta(\sigma) = 1$ *and* $\widetilde{\mathcal{H}}^{(\delta,\sigma)}$ *is a cyclic module, then* (δ, σ) *is a modular pair in involution.*

Proof. As we mentioned the simplicial relations are already held and it remains to check the following extra relations:

$$
\begin{aligned}
\tau_n^{n+1} &= id & (2) \\
\delta_i \tau_n &= \tau_{n-1} \delta_{i-1} & (3) \\
\delta_0 \tau_n &= \delta_n & (4) \\
\sigma_i \tau_n &= \tau_{n+1} \sigma_{i-1} & (5) \\
\sigma_0 \tau_n &= \tau_{n+1}^2 \sigma_n. & (6)
\end{aligned}
$$

To prove equation (2), we first compute τ_n^2 :

$$
\begin{aligned}
\tau_n^2(h_1 \otimes h_2 \otimes ... \otimes h_n) &= \tau(\sum \delta(h_n^2) S_\sigma(h_1^{(1)} h_2^{(1)}...h_n^{(1)}) \otimes h_1^{(2)} \otimes ... \otimes h_{n-1}^{(2)}) \\
&= \sum \delta(h_n^{(2)}) \tau_n(S_\sigma(h_1^{(1)} h_2^{(1)}...h_n^{(1)}) \otimes h_1^{(2)} \otimes ... \otimes h_{n-1}^{(2)}) \\
&= \sum \delta(h_n^{(2)}) \sum \delta((h_{n-1}^{(2)})^{(2)}) S_\sigma(S_\sigma(h_1^{(1)}...h_n^{(1)})^{(1)} (h_1^{(2)})^{(1)}... \\
&\quad (h_{n-1}^{(2)})^{(1)}) \otimes (S_\sigma(h_1^{(1)}...h_n^{(1)})^{(2)} \otimes (h_1^{(2)}))^{(2)} \otimes ... \otimes (h_{n-2}^{(2)})^{(2)} \\
&= \sum \delta(h_n^{(3)}) \delta(h_{n-1}^{(4)}) S_\sigma(\sigma S(h_n^{(2)})...S(h_1^{(2)}) h_1^{(3)}...h_{n-1}^{(3)}) \otimes \\
&\quad S_\sigma(h_1^{(1)} h_2^{(1)}...h_n^{(1)}) \otimes h_1^{(4)}... \otimes h_{n-2}^{(4)} \\
&= \sum \delta(h_n^{(3)}) \delta(h_{n-1}^{(3)}) S_\sigma(\sigma S(h_n^{(2)}) \epsilon(h_{n-1}^{(2)})... \\
&\quad \epsilon(h_1^{(2)})) \otimes S_\sigma(h_1^{(1)}...h_n^{(1)}) \otimes (h_1^{(3)} \otimes h_2^{(3)} \otimes ... \otimes h_{n-2}^{(3)}) \\
&= \sum \delta(h_n^{(3)}) \delta(h_{n-1}^{(2)}) S_\sigma^2(h_n^{(2)}) \otimes S_\sigma(h_1^{(1)}...h_n^{(1)}) \\
&\quad \otimes h_1^{(2)} \otimes h_2^{(2)} \otimes ... \otimes h_{n-2}^{(2)}.
\end{aligned}
$$

By a similar argument, we can deduce

$$
\tau_n^3(h_1 \otimes h_2 \otimes ... \otimes h_n) = \sum \delta(h_n^{(3)}) \delta(h_{n-1}^{(3)}) \delta(h_{n-2}^{(2)})
$$
$$
S_\sigma^2(h_{n-1}^{(2)}) \otimes S_\sigma^2(h_n^{(2)}) \otimes S_\sigma(h_1^{(1)}...h_n^{(1)}) \otimes h_1^{(2)} \otimes h_2^{(2)} \otimes ... \otimes h_{n-3}^{(2)}.
$$

Continuing,

$$
\tau_n^n(h_1 \otimes h_2 \otimes ... \otimes h_n)
$$
$$
= \sum \delta(h_n^{(3)})...\delta(h_2^{(3)}) \delta(h_1^{(2)}) S_\sigma^2(h_2^{(2)}) \otimes ...S_\sigma^2(h_n^{(2)}) \otimes S_\sigma(h_1^{(1)}...h_n^{(1)})
$$

and eventually,

$$
\tau_n^{n+1}(h_1 \otimes h_2 \otimes ... \otimes h_n) =
$$
$$
\sum \delta(h_n^{(3)})...\delta(h_1^{(3)}) \delta(S(h_1^{(1)}))...\delta(S(h_n^{(1)})) S_\sigma^2(h_1^{(2)}) \otimes ... \otimes S_\sigma^2(h_n^{(2)})
$$

$$
= \widetilde{S}_\sigma^2(h_1) \otimes ... \otimes \widetilde{S}_\sigma^2(h_n) = h_1 \otimes h_2 \otimes ... \otimes h_n.
$$

We leave it to the reader to check the remaining equations. To prove the converse it suffices to use just $\tau_1^2 = id$. $\square$

Example 2.1. *Let G be a discrete group and kG be its group algebra over k. It is a cocommutative Hopf algebra with the coproduct, counit and antipode defined by $\Delta(g) = g \otimes g$; $\epsilon(g) = 1$; $S(g) = g^{-1}$. We compute the cyclic module $\widetilde{kG}^{(\epsilon,1)}$. It is obvious that $kG^{\otimes n}$ can be identified with kG^n, the free k module generated by G^n. So we have*

$$\delta_i(g_1, ..., g_n) = \begin{cases} (g_2, ..., g_n) & if \ i = 0 \\ (g_1, ..., g_i g_{i+1}, ... g_n) & 1 \leq i < n \\ (g_1, ..., g_{n-1}) & if \ i = n \end{cases}$$

$$\sigma_i(g_1, ..., g_n) = \begin{cases} (1, g_1, ..., g_n) & if \ i = 0 \\ (g_1, ..., g_i, 1, g_{i+1}, ... g_n) & 1 \leq i \leq n-1 \\ (g_1, ..., g_{n-1}, g_n, 1) & if \ i = n \end{cases}$$

$$\tau(g_1, g_2, ..., g_n) = ((g_1 g_2 ... g_n)^{-1}, g_1, ..., g_{n-1}).$$

It follows that the cyclic module $\widetilde{kG}^{(\epsilon,1)}$ exactly coincides with kBG, the cyclic module associated with the classifying space of G [9].

Corollary 2.1. *Let G be as in the previous example. Then we have*

$$\widetilde{HP}_n^{(\epsilon,1)}(kG) = \lim_{\leftarrow} HC_{n+2i}(kBG) = \begin{cases} \prod_{i \geq 0} H_{2i}(G; k) & n \ even \\ \prod_{i \geq 0} H_{2i+1}(G; k) & n \ odd. \end{cases}$$

3 Hopf Algebra Coaction on an Algebra

In this section we consider a Hopf algebra $\mathcal{H}$ that has a right coaction on an algebra A. In technical terms A is a right comodule algebra i.e., there is a k-linear map $\beta : A \longrightarrow A \otimes \mathcal{H}$ such that the following diagrams commute

$$
\begin{array}{ccc}
A & \xrightarrow{\ \beta\ } & A \otimes \mathcal{H} \\
\beta \downarrow & & \downarrow I_A \otimes \Delta \\
A \otimes \mathcal{H} & \xrightarrow{\beta \otimes I_{\mathcal{H}}} & A \otimes \mathcal{H} \otimes \mathcal{H}
\end{array}
\qquad
\begin{array}{ccc}
A & \xrightarrow{\ \beta\ } & A \otimes \mathcal{H} \\
 & \cong \searrow & \downarrow I_A \otimes \epsilon \\
 & & A \otimes k
\end{array}
$$

and β is a algebra map where the algebra structure of $A \otimes \mathcal{H}$ is the tensor product of the algebras A and $\mathcal{H}$. Similarly one can define a left comodule algebra. In a practical way, we can state the commutativity by

$$\sum \sum a_{(0)} \otimes (a_{(1)})^{(1)} \otimes (a_{(1)})^{(2)} = \sum \sum (a_{(0)})_{(0)} \otimes (a_{(0)})_{(1)} \otimes a_{(1)} \quad (7)$$

$$a = \sum a_{(0)} \epsilon(a_{(1)}) \quad (8)$$

where the above notations mean

$$\beta(a) = \sum a_{(0)} \otimes a_{(1)}$$

$$\Delta(h) = \sum h^{(1)} \otimes h^{(2)}.$$

Definition 3.1. *A linear map, $Tr : A \to k$ is called δ-trace if*

$$Tr(ab) = \sum_{(b)} Tr(ba^{(0)})\delta(a^{(1)}) \qquad\qquad \forall a, b \in A.$$

It is called σ-invariant if for all $a, b \in A$,

$$\sum_{(b)} Tr(a^{(0)}b)\ (a^{(1)}) = \sum_{(a)} Tr(ab^{(0)})S_\sigma(b^{(1)}),$$

or equivalently

$$Tr(a^{(0)})a^{(1)} = Tr(a)\sigma.$$

Let $C_*(A)$ denote the cyclic module of the algebra A.

Proposition 3.1. *Let Tr be a δ-trace on A, which is σ-invariant. Then the following map is a cyclic map:*

$$\gamma : C_n(A) \longrightarrow \widetilde{\mathcal{H}}_n^{(\delta,\sigma)},$$

$$\gamma(a_0 \otimes a_1 \otimes ... \otimes a_n) = \sum Tr(a_0 a_1^{(0)} a_2^{(0)} ... a_n^{(0)})(a_1^{(1)} \otimes ... \otimes a_n^{(1)}). \qquad (9)$$

Proof. We should verify that γ commutes with $\delta_i, \sigma_i, \tau_n$:

$$
\begin{aligned}
\gamma \circ \delta_0(a_0 \otimes a_1 \otimes ... \otimes a_n) &= \gamma(a_0 a_1 \otimes ... \otimes a_n) \\
&= \sum Tr(a_0 a_1 a_2^{(0)} ... a_n^{(0)})(a_2^{(1)} \otimes a_3^{(1)} \otimes ... \otimes a_n^{(1)}) \\
&= \sum Tr(a_0 a_1^{(0)} \epsilon(a_1^{(1)}) a_2^{(0)} ... a_n^{(0)})(a_2^{(1)} \otimes a_3^{(1)} \otimes ... \otimes a_n^{(1)}) \\
&= \sum Tr(a_0 a_1^{(0)} a_2^{(0)} ... a_n^{(0)})(\epsilon(a_1^{(1)}) a_2^{(1)} \otimes a_3^{(1)} \otimes ... \otimes a_n^{(1)}) \\
&= \delta_0 \circ \gamma(a_0 \otimes a_1 \otimes ... \otimes a_n).
\end{aligned}
$$

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For $1 \leq i \leq n-1$, it is obvious that γ commutes with δ_i. For $i = n$ we have

$$
\begin{aligned}
\gamma \circ \delta_n(a_0 \otimes a_1 \otimes ... \otimes a_n) &= \gamma(a_n a_0 \otimes a_1 \otimes ... \otimes a_{n-1}) \\
&= \sum Tr(a_n a_0 a_1^{(0)} a_2^{(0)} ... a_{n-1}^{(0)})(a_1^{(1)} \otimes a_2^{(1)} \otimes ... \otimes a_{n-1}^{(1)}) \\
&= \sum Tr(a_0 a_1^{(0)} a_2^{(0)} ... a_n^{(0)})(\delta(a_n^{(1)}) a_1^{(1)} \otimes a_2^{(1)} \otimes ... \otimes a_{n-1}^{(1)}) \\
&= \delta_n \circ \gamma(a_0 \otimes a_1 \otimes ... \otimes a_n),
\end{aligned}
$$

where we have made use of the δ-trace property of Tr.
We leave it to the reader to check $\gamma \circ \sigma_i = \sigma_i \circ \gamma$. Finally, we show that γ commutes with τ_n:

$$
\begin{aligned}
\gamma \circ \tau_n(a_0 \otimes a_1 \otimes ... \otimes a_n) &= \gamma(a_n \otimes a_0 \otimes ... \otimes a_{n-1}) \\
&= \sum Tr(a_n a_0^{(0)} a_1^{(0)} a_2^{(0)} ... a_{n-1}^{(0)})(a_0^{(1)} \otimes a_1^{(1)} \otimes ... \otimes a_{n-1}^{(1)})
\end{aligned}
$$

and

$$
\begin{aligned}
\tau_n \circ \gamma(a_0 \otimes a_1 \otimes ... \otimes a_n) &= \sum Tr(a_0 a_1^{(0)} a_2^{(0)} ... a_n^{(0)}) \tau(a_1^{(1)} \otimes a_2^{(1)} \otimes ... \otimes a_n^{(1)}) \\
&= \sum Tr(a_0 a_1^{(0)} a_2^{(0)} ... a_n^{(0)})(S_\sigma(a_1^{(1)} a_2^{(1)} ... a_n^{(1)}) \\
&\qquad \otimes a_1^{(2)} \otimes ... \otimes \delta(a_n^{(2)}) a_{n-1}^{(2)}) \\
&= \sum Tr(a_0^{(0)} a_1^{(0)} a_2^{(0)} ... a_n^{(0)}) (a_0^{(1)} \otimes a_1^{(1)} \otimes ... \otimes \delta(a_n^{(1)}) a_{(n-1)}^{(1)}) \\
&= \sum Tr(a_n a_0^{(0)} a_1^{(0)} a_2^{(0)} ... a_{(n-1)}^{(0)})(a_0^{(1)} \otimes a_1^{(1)} \otimes ... \otimes a_{(n-1)}^{(1)}).
\end{aligned}
$$

$\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Corollary 3.1. *Under the conditions of Proposition 2.1,γ induces the following canonical map:*

$$
\gamma_* : \; HC_*(A) \longrightarrow \widetilde{HC}_*^{(\delta,\sigma)}(\mathcal{H}). \tag{10}
$$

Example 3.1. *Let A be $sl_q(2, \mathbb{C})$ and $\mathcal{H}$ be $\mathbb{C}[z, z^{-1}]$. Then $\mathcal{H}$ has a natural coaction on A as follows. If we denote the generators of $sl_q(2, \mathbb{C})$ by a, b, c, d,*

then our coaction is

$$\begin{aligned}
\beta(a) &= a \otimes z \\
\beta(b) &= b \otimes z^{-1} \\
\beta(c) &= c \otimes z \\
\beta(d) &= d \otimes z^{-1}.
\end{aligned}$$

If we consider

$$Tr(x) = \begin{cases} 1 & \text{if } x = a^k d^k \text{ for } k \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

then Tr is a ϵ-trace and it is 1-invariant.

Proposition 3.2. *Let $\mathcal{H}$ be a Hopf algebra with $S_\sigma^2 = id$ for some grouplike element σ. Then the following map is a cyclic map:*

$$\theta : \widetilde{\mathcal{H}}_n^{(\epsilon,\sigma)} \longrightarrow C_n(\mathcal{H}),$$

$$\theta(h_1 \otimes h_2 \otimes \cdots \otimes h_n) = (S_\sigma(h_1^{(1)} h_2^{(1)} \cdots h_n^{(1)}) \otimes h_1^{(2)} \otimes h_2^{(2)} \otimes \cdots \otimes h_n^{(2)}).$$

Here, $C_n(\mathcal{H})$ is the corresponding cyclic module when $\mathcal{H}$ is considered to be an algebra.

Proof. We must show that θ commutes with δ_i, σ_i and τ. In this proof we show $\theta\delta_n = \delta_n\theta$, $\theta\tau = \tau\theta$ and leave the other cases to the reader.
At first, let $h \in \mathcal{H}$, so $h = S_\sigma^2(h) = \sigma S^2(h)\sigma^{-1}$. Therefore $h^{(2)} S_\sigma(h^{(1)}) = \sigma S^2(h^{(2)})S(h^{(1)}) = \sigma\epsilon(h)$. Now,

$$\begin{aligned}
\theta\delta_n(h_1 \otimes h_2 \otimes \cdots \otimes h_n) &= \epsilon(h_n)\theta(h_1 \otimes h_2 \otimes \cdots \otimes h_{n-1}) \\
&= \epsilon(h_n)(S_\sigma(h_1^{(1)} \cdots h_{n-1}^{(1)}) \otimes h_1^{(2)} \otimes h_2^{(2)} \otimes \cdots \otimes h_{n-1}^{(2)}) \\
\delta_n\theta(h_1 \otimes h_2 \otimes \cdots \otimes h_n) &= \delta_n(S_\sigma(h_1^{(1)} \ldots h_n^{(1)}) \otimes h_1^2 \otimes \cdots \otimes h_n^{(2)}) \\
&= (h_n^{(2)} S_\sigma(h_1^{(1)} \ldots h_n^{(1)}) \otimes h_1^{(2)} \otimes \cdots \otimes h_{n-1}^{(2)}) \\
&= \epsilon(h_n)(S_\sigma(h_1^{(1)} \cdots h_{n-1}^{(1)}) \otimes h_1^{(2)} \otimes \cdots \otimes h_n^{(2)}),
\end{aligned}$$

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so that $\theta\delta_n = \delta_n\theta$. Next, we have

$$
\begin{aligned}
\theta\tau(h_1 \otimes h_2 \otimes \cdots \otimes h_n) &= \theta(S_\sigma(h_1^{(1)} \cdots h_n^{(1)}) \otimes h_1^{(2)} \otimes \cdots \otimes \epsilon(h_n^{(2)})h_{n-1}^{(2)}) \\
&= \epsilon(h_n^{(3)})(S_\sigma(S_\sigma(h_1^{(2)} \cdots h_n^{(2)})h_1^{(3)} \cdots h_{n-1}^{(3)}) \\
&\quad \otimes S_\sigma(h_1^{(1)} \cdots h_n^{(1)}) \otimes h_1^{(4)} \otimes \cdots \otimes h_{n-1}^{(4)}) \\
&= \epsilon(h_n^{(3)})(S_\sigma^2(h_n^{(2)}) \otimes S_\sigma(h_1^{(1)} \cdots h_n^{(1)}) \otimes h_1^{(2)} \\
&\quad \otimes h_2^{(2)} \otimes \cdots \otimes h_{n-1}^{(2)}) \\
&= (h_n^{(2)} \otimes S_\sigma(h_1^{(1)} \cdots h_n^{(1)}) \otimes h_1^{(2)} \otimes \cdots \otimes h_{n-1}^{(2)}).
\end{aligned}
$$

On the other hand,

$$
\begin{aligned}
\tau\theta(h_1 \otimes h_2 \otimes \cdots \otimes h_n) &= \tau(S_\sigma(h_1^{(1)} \cdots h_n^{(1)}) \otimes h_1^{(2)} \otimes \cdots \otimes h_n^{(2)}) \\
&= (h_n^{(2)} \otimes S_\sigma(h_1^{(1)} \cdots h_n^{(1)}) \otimes h_1^{(2)} \otimes \cdots \otimes h_{n-1}^{(2)}).
\end{aligned}
$$

$\square$

One knows that any Hopf algebra $\mathcal{H}$ has a right coaction on itself by comultiplication. Let $\mathcal{H}$ have a σ-invariant trace Tr. By Proposition 3.1, we have a map

$$
\gamma : C_n(\mathcal{H}) \longrightarrow \widetilde{\mathcal{H}}_n^{(\epsilon,\sigma)},
$$

$$
\gamma(h_0 \otimes h_1 \otimes \cdots \otimes h_n) = \sum Tr(h_0 h_1^1 \cdots h_n^1)(h_1^2 \otimes \cdots \otimes h_n^2).
$$

Theorem 3.1. *Let $\mathcal{H}$ be a Hopf algebra with $S_\sigma^2 = id$ for some σ, and also, let $\mathcal{H}$ have a σ-invariant trace, Tr, and let $Tr(\sigma)$ be invertible in k. Then, $\widetilde{HC}_n^{(\epsilon,\sigma)}(\mathcal{H})$ is a direct summand in $HC_n(\mathcal{H})$ where $\mathcal{H}$ is considered as an algebra.*

Proof. It can be shown that $\gamma\theta = Tr(\sigma)id$. $\square$

Example 3.2. *Let G be a discrete group and $\mathcal{H} = kG$ be its group algebra over k. Let σ be any central element in G. It is obvious that the following trace satisfies all conditions needed in the previous theorem:*

$$
Tr(x) = \begin{cases} 1 & \text{if } x = \sigma \\ 0 & \text{otherwise.} \end{cases}
$$

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4 Relation with Hopf Algebra Homology

In this section we recall the analogue of group homology for Hopf algebras and relate our cyclic homology, for cocommutative Hopf algebras, to this homology. Let $\mathcal{H}$ be a Hopf algebra and M a left $\mathcal{H}$-module. We define two new modules, the module of invariants and coinvariants, $M^{\mathcal{H}}$ and $M_{\mathcal{H}}$ by :

$$M^{\mathcal{H}} = \{m \in M | hm = \epsilon(h)m \quad \forall\, h \in \mathcal{H}\} \tag{11}$$

$$M_{\mathcal{H}} = M/\{\text{submodule generated by}(hm - \epsilon(h)m)|h \in \mathcal{H}, m \in M\}. \tag{12}$$

In fact, we have two functors $-^{\mathcal{H}}$ and $-_{\mathcal{H}}$ from $\mathcal{H}$-mod to k-mod and by recalling that a trivial $\mathcal{H}$-module is an $\mathcal{H}$ module where $hm = \epsilon(h)m$ for all $m \in M$, and all $h \in \mathcal{H}$, we see that $M^{\mathcal{H}}$ is the biggest trivial submodule of M, and $M_{\mathcal{H}}$ is the biggest quotient module of M that is trivial under the action of $\mathcal{H}$. On the other hand, we have the trivial module functor from k-mod to $\mathcal{H}$-mod with $-_{\mathcal{H}}$ as its right adjoint and $-^{\mathcal{H}}$ its left adjoint. It is obvious that $M_{\mathcal{H}} = k \underset{\mathcal{H}}{\otimes} M$ and $M^{\mathcal{H}} = \hom_{\mathcal{H}}(k, M)$ where k is the trivial $\mathcal{H}$-module. In the following L_* and R^* denote the left and right derived functors.

Definition 4.1. *Let M be an $\mathcal{H}$-module. We define $H_*(\mathcal{H}; M)$ to be $L_*(-^{\mathcal{H}})(M)$ and call them Hopf algebra homology groups of $\mathcal{H}$ with coefficients in M. Using the above notations, we have $H_*(\mathcal{H}; M) \cong Tor_*^{\mathcal{H}}(k, M)$. Similarly, if we define $H^*(\mathcal{H}; M)$ to be $R^*(-_{\mathcal{H}})(M)$, we have $H^*(\mathcal{H}; M) \cong Ext_{\mathcal{H}}^*(k, M)$.*

Example 4.1. *Let $\mathfrak{g}$ be a Lie algebra and $\mathcal{H} = U(\mathfrak{g})$ its enveloping algebra. Then, an $\mathcal{H}$-module is exactly a $\mathfrak{g}$-module and we have $M^{\mathcal{H}} = M^{\mathfrak{g}}$ and $M_{\mathcal{H}} = M_g$. So $H^*(\mathcal{H}; M) = H^*(\mathfrak{g}; M)$ is the Lie algebra cohomology and, $H_*(\mathcal{H}; M) = H_*(\mathfrak{g}; M)$. Similarly if $\mathcal{H} = kG$ is the group algebra of a (discrete) group G, then $H_*(\mathcal{H}; M) \cong H_*(G; M)$ is the group homology and $H^*(\mathcal{H}; M) \cong H^*(G; M)$.*

For every simplicial object M one can define its path space, EM, where $EM_n = M_{n+1}$ and its nth face is $(n+1)^{th}$ face of M and the same for degeneracies. So if one denotes the path space of $\widetilde{\mathcal{H}}^{(\delta,\sigma)}$ by $E\mathcal{H}$, its simplicial structure is:

$$\delta_i(h_0 \otimes h_1 \otimes ... \otimes h_n) = h_0 \otimes h_1 \otimes ... \otimes h_i h_{i+1} \otimes ... \otimes h_n \ \ 0 \le i \le n-1$$
$$\delta_n(h_0 \otimes h_1 \otimes ... \otimes h_n) = \delta(h_n)h_0 \otimes h_1 \otimes ... \otimes h_{n-1}$$
$$\sigma_i(h_0 \otimes h_1 \otimes ... \otimes h_n) = h_0 \otimes h_1... \otimes h_i \otimes 1 \otimes h_{i+1}... \otimes h_n \ \ 0 \le i \le n-1$$
$$\sigma_n(h_0 \otimes h_1 \otimes ... \otimes h_n) = h_0 \otimes h_1 \otimes ... \otimes 1.$$

It is easy to verify that $E\mathcal{H}$ is a simplicial k-module, contractible and also a free resolution for k via $\delta : E\mathcal{H} \longrightarrow k$.

Now let M_* be a chain complex of $\mathcal{H}$-modules. We denote the hyper derived functors $\mathbb{L}(-_\mathcal{H})(M_*)$ by $\mathbb{H}(\mathcal{H}; M_*)$ and $\mathbb{R}(-^\mathcal{H})(M^*)$ by $\mathbb{H}(\mathcal{H}; M^*)$, where M^* is a cochain complex of $\mathcal{H}$-modules.

Lemma 4.1. *If $\mathcal{H}$ is a cocommutative Hopf algebra then $E\mathcal{H}$ is a cyclic k-module with*

$$t_n(h_0 \otimes \cdots \otimes h_n) = \sum (h_0 h_1^{(1)} \ldots h_n^{(1)}) \otimes S(h_1^{(2)} \ldots h_n^{(2)}) \otimes h_1^{(3)} \otimes \cdots \otimes h_{n-1}^{(3)}.$$

Proof. As always it is needed to verify the relations 2.2,...,2.6. We only check 2.2 and leave the others to the reader.

$$t_n^2(h_0 \otimes \cdots \otimes h_n) = t_n(\sum (h_0 h_1^{(1)} \ldots h_n^{(1)}) \otimes S(h_1^{(2)} \ldots h_n^{(2)}) \otimes h_1^{(3)} \otimes \cdots \otimes h_{n-1}^{(3)})$$

$$= \sum (h_0 h_1^{(1)} h_2^{(1)} \ldots h_n^{(1)} S(h_1^{(4)} h_2^{(4)} \ldots h_n^{(4)}) h_1^{(5)} \cdots \otimes h_{n-1}^{(5)} \otimes$$

$$S(S(h_1^{(3)} h_2^{(3)} \ldots h_n^{(1)}) h_1^{(6)} \ldots h_{n-1}^{(6)}) \otimes S(h_1^{(2)} \ldots h_n^{(2)}) \otimes h_1^{(7)} \otimes \cdots \otimes h_{n-2}^{(7)}$$

$$= \sum (h_0 h_1^{(1)} \ldots h_{n-1}^{(1)} \otimes h_n^{(1)} \otimes S(h_1^{(2)} h_2^{(2)} \ldots h_n^{(2)}) \otimes h_1^{(3)} \otimes \cdots \otimes h_{n-2}^{(3)}.$$

By a similar argument we get,

$$t_n^n(h_0 \otimes \cdots \otimes h_n) = \sum h_0 h_1^{(1)} \otimes h_2^{(3)} \cdots \otimes h_n^{(3)} \otimes S(h_1^{(2)} \ldots h_n^{(2)}),$$

and finally we have

$$t_n^{n+1}(h_0 \otimes \cdots \otimes h_n) = h_0 \otimes \cdots \otimes h_n.$$

$\square$

From now on we denote $B\mathcal{H}$ for $\widetilde{\mathcal{H}}^{(\epsilon,1)}$.

Lemma 4.2. *The projection $\pi : E\mathcal{H} \longrightarrow B\mathcal{H}$ where $\pi(h_o \otimes \cdots \otimes h_n) = (\epsilon(h_o) h_1 \otimes \cdots \otimes h_n)$ is a simplicial map and, if $\mathcal{H}$ is cocommutative, then, π is a cyclic map.*

Proof. We leave it to the reader the first part of proof and just prove the second part. We must verify that the following diagram is commutative.

$$
\begin{array}{ccc}
E\mathcal{H}_n & \xrightarrow{\ \pi\ } & B\mathcal{H}_n \\
{\scriptstyle t}\big\downarrow & & \big\downarrow{\scriptstyle \tau} \\
E\mathcal{H}_n & \xrightarrow{\ \pi\ } & B\mathcal{H}_n.
\end{array}
$$

We have

$$
\tau \circ \pi(h_0 \otimes \cdots \otimes h_n) = \tau(\epsilon(h_0)h_1 \otimes \cdots \otimes h_n)
$$

$$
= \epsilon(h_0) \sum (S(h_1^{(1)} h_2^{(1)} \ldots h_{n-1}^{(1)} h_n) \otimes h_1^{(2)} \otimes \cdots \otimes h_{n-1}^{(2)}),
$$

and

$$
\pi \circ t(h_0 \otimes \cdots \otimes h_n) = \pi(\sum (h_0 h_1^{(1)} \ldots h_n^1) \otimes S(h_1^{(2)} h_2^{(2)} \ldots h_n^{(2)}) \otimes h_1^{(3)} \otimes \ldots h_{n-1}^{(3)}
$$

$$
= \sum (\epsilon(h_0 h_1^{(1)} h_2^{(1)} \ldots h_n^{(1)}) S(h_1^{(2)} \ldots h_n^{(2)}) \otimes h_1^{(3)} \otimes \ldots h_{n-1}^{(3)})
$$

$$
= \epsilon(h_0) \sum (S(h_1^{(1)} \ldots h_{n-1}^{(1)} h_n) \otimes h_1^{(2)} \otimes \cdots \otimes h_{n-1}^{(2)}).
$$

$\square$

It is obvious that $E\mathcal{H}$ is an $\mathcal{H}$-module via $h(h_0 \otimes \cdots \otimes h_n) = (hh_0 \otimes \cdots \otimes h_n)$, and the relation between $E\mathcal{H}$ and $B\mathcal{H}$ becomes $B\mathcal{H} = k \otimes_{\mathcal{H}} E\mathcal{H}$.

Next, we prove a theorem which computes the cyclic homology of co-commutative Hopf algebras. For $\mathcal{H} = kG$, our result reduces to Karoubi's theorem [6].

Theorem 4.1. *If $\mathcal{H}$ is a cocommutative Hopf algebra then*

$$
\widetilde{HC}_n^{(\epsilon,1)}(\mathcal{H}) = \bigoplus_{i \geq 0} H_{n-2i}(\mathcal{H}; k).
$$

Proof. By the above remark we have $CC_{**}(B\mathcal{H}) \cong k \otimes_{\mathcal{H}} CC_{**}(E\mathcal{H})$ where CC_{**} denotes cyclic double complex [9]. Since $E\mathcal{H}$ is contractible and $\epsilon : E\mathcal{H} \longrightarrow k$ is a Homotopy equivalence the double complex $CC_{**}(E\mathcal{H})$ is a resolution for k_*, where $k*$ is

$$
k \leftarrow 0 \leftarrow k \leftarrow 0 \ldots.
$$

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On the other hand,

$$\widetilde{HC}_n^{(\epsilon,1)}(\mathcal{H}) = H_n(TotCC_{**}(B\mathcal{H})) =$$

$$H_n(TotCC_{**}(k \otimes_{\mathcal{H}} E\mathcal{H}) = \mathbb{H}(\mathcal{H}; k_*).$$

So, to complete the proof it suffices to compute $\mathbb{H}(\mathcal{H}; k_*)$. But by finding a Cartan-Eilenberg resolution for k_*, we have

$$\mathbb{H}(\mathcal{H}; k_*) = \bigoplus_{i \geq 0} H_{n-2i}(\mathcal{H}; k).$$

$\square$

Example 4.2. *Let $\mathfrak{g}$ be a Lie algebra over k and $U(\mathfrak{g})$ be its enveloping algebra. One knows that $H_n(U(\mathfrak{g}); k) = H_n(\mathfrak{g}; k)$, [9] so by Theorem 4.1 we have*

$$\widetilde{HC}_n^{(\epsilon,\sigma)}(U(\mathfrak{g})) = \bigoplus_{k \geq 0} H_{n-2k}(\mathfrak{g}; k).$$

Now let $\mathcal{H}$ be a Hopf algebra and M be an $\mathcal{H}$-bimodule. We can convert M to a new left $\mathcal{H}$-module, $\widetilde{M} = M$, where the action of $\mathcal{H}$ is

$$h \blacktriangleright m = h^{(2)} m S(h^1).$$

Proposition 4.1. *(Mac Lane isomorphism for Hopf algebras)*
Under the above hypotheses there is a canonical isomorphism

$$\theta_* : H_n(\mathcal{H}, M) \cong H_n(\mathcal{H}; \widetilde{M}).$$

Proof. If $C_n(\mathcal{H}; \widetilde{M}) = \mathcal{H}^{\otimes n} \otimes \widetilde{M}$ then it is obvious that $C_n(\mathcal{H}; \widetilde{M})$ is a simplicial module by the following faces and degeneracies:

$$\delta_0(h_1 \otimes \cdots \otimes h_n \otimes m) = (\epsilon(h_1)h_2 \cdots \otimes h_n \otimes m)$$
$$\delta_i(h_1 \otimes \cdots \otimes h_n \otimes m) = (h_1 \otimes \cdots \otimes h_i h_{i+1} \otimes \cdots \otimes h_n \otimes m) \qquad 1 \leq i \leq n-1$$
$$\delta_n(h_1 \otimes \cdots \otimes h_n \otimes m) = (h_1 \otimes \cdots \otimes h_{n-1} \otimes h_n \blacktriangleright m)$$
$$\sigma_i(h_1 \otimes \cdots \otimes h_n \otimes m) = (h_1 \otimes \cdots \otimes h_i \otimes 1 \otimes h_{i+1} \cdots \otimes h_n \otimes m) \qquad 0 \leq i \leq n.$$

Now, $H_*(\mathcal{H}; \widetilde{M})$ can be computed by the above complex. Let

$$\theta(m \otimes h_1 \otimes \cdots \otimes h_n) = \sum (h_1^{(2)} \otimes \cdots \otimes h_n^{(2)} \otimes m h_1^{(1)} h_2^{(1)} \ldots h_n^{(1)}).$$

We leave to the reader to show θ is a simplicial map and in fact, θ is a simplicial isomorphism by the following inverse map

$$\theta^{-1}(h_1 \otimes \cdots \otimes h_n \otimes m) = \sum (m S(h_1^{(1)} h_2^{(1)} \ldots h_n^{(1)}) \otimes h_1^{(2)} \otimes \cdots \otimes h_n^{(2)}).$$

$\square$

Let k be the trivial $\mathcal{H}$-module. Then $\widetilde{k}$ is also a trivial $\mathcal{H}$-module. So we have,

$$H_n(\mathcal{H}, k) = H_n(\mathcal{H}; k),$$

where the right hand side is the Hochschild homology of $\mathcal{H}$ with trivial coefficients via ϵ for both left and right action of $\mathcal{H}$ on k. The left hand side is the Hopf algebra homology of $\mathcal{H}$ via ϵ.

Corollary 4.1. *If $\mathcal{H}$ is a cocommutative Hopf algebra then,*

$$\widetilde{HC}_n^{(\epsilon,1)}(\mathcal{H}) = \bigoplus_{k \geq 0} H_{n-2k}(\mathcal{H}, k).$$

Example 4.3. *Let V be a k-module and $T(V)$ be its tensor algebra .Then $T(V)$ is a cocommutative Hopf algebra. We have $H_0(T(V), k) = k$, $H_1(T(V), k) = V$ and the other homology groups are zero so,*

$$\widetilde{HC}_n^{(\epsilon,1)}(T(V)) = k \qquad \text{if } n \text{ is even}$$

$$\widetilde{HC}_n^{(\epsilon,1)}(T(V)) = V \qquad \text{if } n \text{ is odd.}$$

Remark (Connes-Moscovici cyclic cohomology of commutative Hopf algebras).
Using methods similar to the above, one can compute Connes-Moscovici periodic cyclic cohomology $HP^*_{(\epsilon,1)}(\mathcal{H})$ of commutative Hopf algebras. Since proofs are similar we only indicate the main steps.

Proposition 4.2. *Let $\mathcal{H}$ be a commutative Hopf algebra. Let $(E\mathcal{H})_n = \mathcal{H}^{\otimes n+1}$, $n \geq 0$. Then the following operators define a cocyclic module structure on $E\mathcal{H}$:*

$$
\begin{aligned}
d_i(h_0 \otimes \cdots \otimes h_n) &= h_0 \otimes \cdots \otimes \Delta(h_i) \otimes \cdots \otimes h_n \\
d_{n+1}(h_0 \otimes \cdots \otimes h_n) &= h_0 \otimes \cdots \otimes h_n \otimes 1 \\
s_i(h_0 \otimes \cdots \otimes h_n) &= h_0 \otimes \cdots \otimes \epsilon(h_i) \otimes \cdots \otimes h_n \\
\tau(h_0 \otimes \cdots \otimes h_n) &= h_0^{(1)} \otimes h_0^{(2)} S(h_1^{(n-1)}) h_2 \otimes \cdots \otimes h_0^{(n)} S(h_1^{(1)}) h_n.
\end{aligned}
$$

$\square$

Let $\mathcal{H}_{(\epsilon,\delta)}$ denote the Connes-Moscovici cocyclic module of $\mathcal{H}$ for the modular pair $(\epsilon, 1)$.

Proposition 4.3. *The following map is a morphism of cocyclic modules, $\psi : \mathcal{H}^{(\epsilon,1)} \longrightarrow E\mathcal{H}$,*

$$
\psi(h_1 \otimes \cdots \otimes h_n) = 1 \otimes h_1 \otimes \cdots \otimes h_n.
$$

$\square$

Using the above two propositions and dualizing the above method to prove a similar result for cocommutative Hopf algebras, one can prove:

Theorem 4.2. *Let $\mathcal{H}$ be a commutative Hopf algebra. Its periodic cyclic cohomology in the sense of Connes-Moscovici [1, 2] is given by*

$$
HP^n_{(\epsilon,1)}(\mathcal{H}) = \bigoplus_{i=n \ (mod\ 2)} H^i(\mathcal{H}, k)
$$

$\square$

For example, if $\mathcal{H} = k[G]$ is the algebra of regular functions on an affine algebraic group G, then the coalgebra complex of $\mathcal{H} = k[G]$ is isomorphic to the group cohomology complex, with trivial coefficient, where instead of arbitrary cochains one uses regular functions $G \times G \times \cdots \times G \to k$. Denote this cohomology by $H^i(G, k)$. It follows that

$$
HP^n_{(\epsilon,1)}(k[G]) = \bigoplus_{i=n \ (mod\ 2)} H^i(G; k).
$$

For $k = \mathbb{R}$, this gives an alternative proof of Prop.4 and Remark 5 in [3].

5   The Cyclic Homology of $A(SL_q(2,k))$ and $U_q(sl(2,k))$

In this section we compute our cyclic homology theory for $A(SL_q(2,k))$, the quantized algebra of functions on the quantum group $SL_q(2,k)$ and also for the quantized universal enveloping algebra $U_q(sl(2,k))$.

Let k be a field of characteristic zero and $q \in k$, $q \neq 0$ and q not a root of unity. The Hopf algebra $\mathcal{H} = A(SL_q(2,k))$ is defined as follows. As an algebra it is generated by symbols x, u, v, y, with the following relations:

$$ux = qxu, \quad vx = qxv, \quad yu = quy, \quad yv = qvy,$$

$$uv = vu, \quad xy - q^{-1}uv = yx - quv = 1.$$

The coproduct, counit and antipode of $\mathcal{H}$ are defined by

$$\Delta(x) = x \otimes x + u \otimes v, \quad \Delta(u) = x \otimes u + u \otimes y,$$

$$\Delta(v) = v \otimes x + y \otimes v, \quad \Delta(y) = v \otimes u + y \otimes y,$$

$$\epsilon(x) = \epsilon(y) = 1, \quad \epsilon(u) = \epsilon(v) = 0,$$

$$S(x) = y, \quad S(y) = x, \quad S(u) = -qu, \quad S(v) = -q^{-1}v.$$

For more details about $\mathcal{H}$ we refer to [7]. Because $S^2 \neq id$, to define our cyclic structure we need a modular pair (σ, δ) in involution. Let δ be as follows:

$$\delta(x) = q, \quad \delta(u) = 0, \quad \delta(v) = 0, \quad \delta(y) = q^{-1}.$$

And $\sigma = 1$. Then we have $\widetilde{S}^2 = id$.

For Computing cyclic homology we should at first compute the Hochschild homology $H_*(\mathcal{H}, k)$ where k is a $\mathcal{H}$-bimodule via δ, ϵ for left and right action of $\mathcal{H}$ respectively.
One knows $H_*(\mathcal{H}, k) = Tor_*^{\mathcal{H}^e}(\mathcal{H}, k)$, where $\mathcal{H}^e = \mathcal{H} \otimes \mathcal{H}^{op}$. So we need a resolution for k, or $\mathcal{H}$ as $\mathcal{H}^e$-module. We take advantage of the free resolution for $\mathcal{H}$ in [8].

The explicit free resolution of A as $B = \mathcal{H} \otimes \mathcal{H}^o$-module is:

$$\cdots \to M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{\mu} \mathcal{H},$$

where μ is the augmentation and M_*, $* \geq 0$ is a family of left B-module with their free rank over B given by

$$rank(M_0) = 1$$
$$rank(M_1) = 4$$
$$rank(M_2) = 7$$
$$rank(M_*) = 8, \ * \geq 3.$$

We give the B-linear differential mapping $d_* : M_* \longrightarrow M_{*-1}$, $* \geq 0$ in terms of their basis over B. We next give the formulas whose B-linear extensions determine the differential d_*, $* > 0$ together with description of the B-basis at the same time:
$d_1 : M_1 \longrightarrow M_0 = B$
is given by

$$d_1(1 \otimes 1 \otimes e_v) = v \otimes 1 - 1 \otimes v,$$
$$d_1(1 \otimes 1 \otimes e_u) = u \otimes 1 - 1 \otimes u,$$
$$d_1(1 \otimes 1 \otimes e_x) = x \otimes 1 - 1 \otimes x,$$
$$d_1(1 \otimes 1 \otimes e_y) = y \otimes 1 - 1 \otimes y,$$

where e_x, e_y, e_u, e_v form a B-basis for M_1.
$d_2 : M_2 \longrightarrow M_1$

$$
\begin{aligned}
d_2(1 \otimes 1 \otimes (e_v \wedge a_u)) &= (v \otimes 1 - 1 \otimes v) \otimes e_u - (u \otimes 1 - 1 \otimes u) \otimes e_v, \\
d_2(1 \otimes 1 \otimes (e_v \wedge a_x)) &= (v \otimes 1 - 1 \otimes qv) \otimes e_x - (qx \otimes 1 - 1 \otimes x) \otimes e_v, \\
d_2(1 \otimes 1 \otimes (e_v \wedge a_y)) &= (qv \otimes 1 - 1 \otimes v) \otimes e_y - (y \otimes 1 - 1 \otimes qy) \otimes e_v, \\
d_2(1 \otimes 1 \otimes (e_u \wedge a_x)) &= (u \otimes 1 - 1 \otimes qu) \otimes e_x - (qx \otimes 1 - 1 \otimes x) \otimes e_u, \\
d_2(1 \otimes 1 \otimes (e_u \wedge a_y)) &= (qu \otimes 1 - 1 \otimes u) \otimes e_y - (y \otimes 1 - 1 \otimes qy) \otimes e_u, \\
d_2(1 \otimes 1 \otimes \vartheta_S^{(1)}) &= y \otimes 1 \otimes e_x + 1 \otimes x \otimes e_y - qu \otimes 1 \otimes e_v - 1 \otimes qv \otimes e_u, \\
d_2(1 \otimes 1 \otimes \vartheta_T^{(1)}) &= 1 \otimes y \otimes e_x + x \otimes 1 \otimes e_y - q^{-1}u \otimes 1 \otimes e_v - 1 \otimes q^{-1}v \otimes e_u,
\end{aligned}
$$

where $\vartheta_S^{(1)}$, $\vartheta_T^{(1)}$, $e_u \wedge e_x$, $e_v \wedge e_x$, $e_v \wedge e_y$, and $e_v \wedge e_u$, form a B-basis in M_2.
$d_3 : M_3 \longrightarrow M_2$

is given by

$$d_3(1 \otimes 1 \otimes (e_v \wedge e_u \wedge e_x)) = (v \otimes 1 - 1 \otimes qv) \otimes (e_u \wedge e_x) -$$
$$- (u \otimes 1 - 1 \otimes qu) \otimes (e_v \wedge e_x) + (q^2 x \otimes 1 - 1 \otimes x) \otimes (e_v \wedge e_u),$$

$$d_3(1 \otimes 1 \otimes (e_v \wedge e_u \wedge e_y)) = (qv \otimes 1 - 1 \otimes v) \otimes (e_u \wedge e_y) -$$
$$- (qu \otimes 1 - 1 \otimes u) \otimes (e_v \wedge e_y) + (y \otimes 1 - 1 \otimes q^2 y) \otimes (e_v \wedge e_u),$$

$$d_3(1 \otimes 1 \otimes (e_v \wedge \vartheta_S^{(1)})) = (v \otimes 1 - 1 \otimes v) \otimes \vartheta_S^{(1)} - q^{-1} y \otimes 1 \otimes (e_v \wedge e_x) -$$
$$- 1 \otimes q^{-1} x \otimes (e_v \wedge e_y) + 1 \otimes qv \otimes (e_v \wedge e_u),$$

$$d_3(1 \otimes 1 \otimes (e_v \wedge \vartheta_T^{(1)})) = (v \otimes 1 - 1 \otimes v) \otimes \vartheta_T^{(1)} - 1 \otimes y \otimes (e_v \wedge e_x) -$$
$$- x \otimes 1 \otimes (e_v \wedge e_y) + 1 \otimes q^{-1} v \otimes (e_v \wedge e_u),$$

$$d_3(1 \otimes 1 \otimes (e_u \wedge \vartheta_S^{(1)})) = (u \otimes 1 - 1 \otimes u) \otimes \vartheta_S^{(1)} - q^{-1} y \otimes 1 \otimes (e_u \wedge e_x) -$$
$$- 1 \otimes q^{-1} x \otimes (e_u \wedge e_y) - qu \otimes 1 \otimes (e_v \wedge e_u),$$

$$d_3(1 \otimes 1 \otimes (e_u \wedge \vartheta_T^{(1)})) = (u \otimes 1 - 1 \otimes u) \otimes \vartheta_T^{(1)} - 1 \otimes y \otimes (e_u \wedge e_x) -$$
$$- x \otimes 1 \otimes (e_u \wedge e_y) - q^{-1} u \otimes 1 \otimes (e_v \wedge e_u),$$

$$d_3(1 \otimes 1 \otimes (e_x \wedge \vartheta_S^{(1)})) = x \otimes 1 \otimes \vartheta_S^{(1)} - 1 \otimes x \otimes \vartheta_T^{(1)} -$$
$$- q^{-1} u \otimes 1 \otimes (e_v \wedge e_x) - 1 \otimes v \otimes (e_u \wedge e_x)$$

$$d_3(1 \otimes 1 \otimes (e_y \wedge \vartheta_T^{(1)})) = y \otimes 1 \otimes \vartheta_T^{(1)} - 1 \otimes y \otimes \vartheta_S^{(1)} -$$
$$- u \otimes 1 \otimes (e_v \wedge e_y) - 1 \otimes q^{-1} v \otimes (e_u \wedge e_y),$$

where $e_x \wedge \vartheta_S^{(1)}$, $e_y \wedge \vartheta_T^{(1)}$, $e_u \wedge \vartheta_S^{(1)}$, $e_u \wedge \vartheta_T^{(1)}$, $e_v \wedge \vartheta_S^{(1)}$, $e_v \wedge \vartheta_T^{(1)}$, $e_v \wedge e_u \wedge e_x$, and $e_v \wedge e_u \wedge e_y$ form a B- basis in M_3.

$d_{2p+4} : M_{2p+4} \longrightarrow M_{2p+3}$, $p \geq 0$
is given by

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$$d_{2p+4}(1 \otimes 1 \otimes (e_v \wedge e_u \wedge \vartheta_S^{(p+1)})) = (v \otimes 1 - 1 \otimes v) \otimes (e_u \wedge \vartheta_S^{(p+1)}) -$$
$$-(u \otimes 1 - 1 \otimes u) \otimes (e_v \wedge \vartheta_S^{(p+1)}) + q^{-2}y \otimes 1 \otimes (e_v \wedge e_u \wedge e_x \wedge \vartheta_S^{(p)}) + 1 \otimes q^{-2}x \otimes (e_v \wedge e_u \wedge e_y \wedge \vartheta_T^{(p)}),$$

$$d_{2p+4}(1 \otimes 1 \otimes (e_v \wedge e_u \wedge \vartheta_T^{(p+1)})) = (v \otimes 1 - 1 \otimes v) \otimes (e_u \wedge \vartheta_T^{(p+1)}) -$$
$$-(u \otimes 1 - 1 \otimes u) \otimes (e_v \wedge \vartheta_T^{(p+1)}) + 1 \otimes y \otimes (e_v \wedge e_u \wedge e_x \wedge \vartheta_S^{(p)}) + x \otimes 1 (e_v \wedge e_u \wedge e_y \wedge \vartheta_T^{(p)}),$$

$$d_{2p+4}(1 \otimes 1 \otimes (e_v \wedge e_x \wedge \vartheta_S^{(p+1)})) = (v \otimes 1 - 1 \otimes qv) \otimes (e_x \wedge \vartheta_S^{(p+1)}) -$$
$$- qx \otimes 1 \otimes (e_v \wedge \vartheta_S^{(p+1)}) + 1 \otimes x \otimes (e_v \wedge \vartheta_T^{(p+1)}) + 1 \otimes v \otimes (e_v \wedge e_u \wedge e_x \wedge \vartheta_S^{(p)}),$$

$$d_{2p+4}(1 \otimes 1 \otimes (e_v \wedge e_y \wedge \vartheta_T^{(p+1)})) = (qv \otimes 1 - 1 \otimes v) \otimes (e_y \wedge \vartheta_T^{(p+1)}) -$$
$$- y \otimes 1 \otimes (e_v \wedge \vartheta_T^{(p+1)}) + 1 \otimes qy \otimes (e_v \wedge \vartheta_S^{(p+1)}) + 1 \otimes q^{-1}v \otimes (e_v \wedge e_u \wedge e_y \wedge \vartheta_T^{(p)}),$$

$$d_{2p+4}(1 \otimes 1 \otimes (e_u \wedge e_x \wedge \vartheta_S^{(p+1)})) = (u \otimes 1 - 1 \otimes qu) \otimes (e_x \wedge \vartheta_S^{(p+1)}) -$$
$$- qx \otimes 1 \otimes (e_u \wedge \vartheta_S^{(p+1)}) + 1 \otimes x \otimes (e_u \wedge \vartheta_T^{(p+1)}) - q^{-1}u \otimes 1 \otimes (e_v \wedge e_u \wedge e_x \wedge \vartheta_S^{(p)}),$$

$$d_{2p+4}(1 \otimes 1 \otimes (e_u \wedge e_y \wedge \vartheta_T^{(p+1)})) = (qu \otimes 1 - 1 \otimes u) \otimes (e_y \wedge \vartheta_T^{(p+1)}) -$$
$$- y \otimes 1 \otimes (e_u \wedge \vartheta_T^{(p+1)}) + 1 \otimes qy \otimes (e_u \wedge \vartheta_S^{(p+1)}) - u \otimes 1 \otimes (e_v \wedge e_u \wedge e_y \wedge \vartheta_T^{(p)}),$$

$$d_{2p+4}(1 \otimes 1 \otimes \vartheta_S^{(p+2)}) = y \otimes 1 \otimes (e_x \wedge \vartheta_S^{(p+1)}) + 1 \otimes x \otimes (e_y \wedge \vartheta_T^{(p+1)}) -$$
$$- qu \otimes 1 \otimes (e_v \wedge \vartheta_S^{(p+1)}) - 1 \otimes qv \otimes (e_u \wedge \vartheta_T^{(p+1)}),$$

$$d_{2p+4}(1 \otimes 1 \otimes \vartheta_T^{(p+2)}) = 1 \otimes y \otimes (e_x \wedge \vartheta_S^{(p+1)}) + x \otimes 1 \otimes (e_y \wedge \vartheta_T^{(p+1)}) -$$
$$- q^{-1}u \otimes 1 \otimes (e_v \wedge \vartheta_T^{(p+1)}) - 1 \otimes q^{-1}v \otimes (e_u \wedge \vartheta_T^{(p+1)}),$$

where $\omega \wedge \vartheta_*^{(0)}$ is identified with ω for

$$\omega = e_v \wedge e_u \wedge e_x, \ * = S \ \text{or} \ \omega = e_v \wedge e_u \wedge e_y, \ * = T,$$

and where $\vartheta_T^{(p+2)}$, $\vartheta_S^{(p+2)}$, $e_u \wedge e_x \wedge \vartheta_S^{(p+1)}$, $e_u \wedge e_y \wedge \vartheta_T^{(p+1)}$, $e_v \wedge e_x \wedge \vartheta_S^{(p+1)}$, $e_v \wedge e_y \wedge \vartheta_T^{(p+1)}$, $e_v \wedge e_u \wedge \vartheta_T^{(p+1)}$, and $e_v \wedge e_u \wedge \vartheta_S^{(p+1)}$ form a B-basis in M_{2p+4}.

$d_{2p+3} : M_{2p+3} \longrightarrow M_{2p+2}$, $p > 0$
is given by

$$d_{2p+3}(1 \otimes 1 \otimes (e_v \wedge e_u \wedge e_x \wedge \vartheta_S^{(p)})) = (v \otimes 1 - 1 \otimes qv) \otimes (e_v \wedge e_x \wedge \vartheta_S^{(p)}) -$$
$$-(u \otimes 1 - 1 \otimes qu) \otimes (e_v \wedge e_x \wedge \vartheta_S^{(p)}) + q^2 x \otimes 1 \otimes (e_v \wedge e_u \wedge \vartheta_S^{(p)}) - 1 \otimes x \otimes (e_v \wedge e_u \wedge \vartheta_T^{(p)}),$$

$$d_{2p+3}(1 \otimes 1 \otimes (e_v \wedge e_u \wedge e_y \wedge \vartheta_T^{(p)})) = (qv \otimes 1 - 1 \otimes v) \otimes (e_u \wedge e_y \wedge \vartheta_T^{(p)}) -$$
$$-(qu \otimes 1 - 1 \otimes u) \otimes (e_v \wedge e_y \wedge \vartheta_T^{(p)}) + y \otimes 1 \otimes (e_v \wedge e_u \wedge \vartheta_T^{(p)}) - 1 \otimes q^2 y \otimes (e_v \wedge e_u \wedge \vartheta_S^{(p)}),$$

$$d_{2p+3}(1 \otimes 1 \otimes (e_v \wedge \vartheta_S^{(p+1)})) = (v \otimes 1 - 1 \otimes v) \otimes \vartheta_S^{(p+1)} - q^{-1} y \otimes 1 \otimes (e_v \wedge e_x \wedge \vartheta_S^{(p)}) -$$
$$- 1 \otimes q^{-1} x \otimes (e_v \wedge e_y \wedge \vartheta_T^{(p)}) + 1 \otimes qv \otimes (e_v \wedge e_u \wedge \vartheta_S^{(p)}),$$

$$d_{2p+3}(1 \otimes 1 \otimes (e_v \wedge \vartheta_T^{(p+1)})) = (v \otimes 1 - 1 \otimes v) \otimes \vartheta_T^{(p+1)} - 1 \otimes y \otimes (e_v \wedge e_x \wedge \vartheta_S^{(p)}) -$$
$$- x \otimes 1 \otimes (e_v \wedge e_y \wedge \vartheta_T^{(p)}) + 1 \otimes q^{-1} v \otimes (e_v \wedge e_u \wedge \vartheta_T^{(p)}),$$

$$d_{2p+3}(1 \otimes 1 \otimes (e_u \wedge \vartheta_S^{(p+1)})) = (u \otimes 1 - 1 \otimes u) \otimes \vartheta_S^{(p+1)} - q^{-1} y \otimes 1 \otimes (e_u \wedge e_x \wedge \vartheta_S^{(p)}) -$$
$$- 1 \otimes q^{-1} x \otimes (e_u \wedge e_y \wedge \vartheta_T^{(p)}) - qu \otimes 1 \otimes (e_v \wedge e_u \wedge \vartheta_S^{(p)}),$$

$$d_{2p+3}(1 \otimes 1 \otimes (e_u \wedge \vartheta_T^{(p+1)})) = (u \otimes 1 - 1 \otimes u) \otimes \vartheta_T^{(p+1)} - 1 \otimes y \otimes (e_u \wedge e_x \wedge \vartheta_S^{(p)}) -$$
$$- x \otimes 1 \otimes (e_u \wedge e_y \wedge \vartheta_T^{(p)}) + q^{-1} u \otimes 1 \otimes (e_v \wedge e_u \wedge \vartheta_T^{(p)}),$$

$$d_{2p+3}(1 \otimes 1 \otimes (e_x \wedge \vartheta_S^{(p+1)})) = x \otimes 1 \otimes \vartheta_S^{(p+1)} - 1 \otimes x \otimes \vartheta_T^{(p+1)} -$$
$$- q^{-1} u \otimes 1 \otimes (e_v \wedge e_x \wedge \vartheta_S^{(p)}) - 1 \otimes v \otimes (e_u \wedge e_x \wedge \vartheta_S^{(p)}),$$

$$d_{2p+3}(1 \otimes 1 \otimes (e_y \wedge \vartheta_T^{(p+1)})) = y \otimes 1 \otimes \vartheta_T^{(p+1)} - 1 \otimes y \otimes \vartheta_S^{(p+1)} -$$
$$- u \otimes 1 \otimes (e_v \wedge e_y \wedge \vartheta_T^{(p)}) - 1 \otimes q^{-1} v \otimes (e_u \wedge e_y \wedge \vartheta_T^{(p)}),$$

where $e_x \wedge \vartheta_S^{(p+1)}$, $e_y \wedge \vartheta_T^{(p+1)}$, $e_u \wedge \vartheta_T^{(p+1)}$, $e_u \wedge \vartheta_S^{(p+1)}$, $e_v \wedge \vartheta_T^{(p+1)}$, $e_v \wedge \vartheta_S^{(p+1)}$, $e_v \wedge e_u \wedge e_x \wedge \vartheta_S^{(p)}$, and $e_v \wedge e_u \wedge e_y \wedge \vartheta_T^{(p)}$ form a B-basis in M_{2p+3}.

By a lengthy computation $H_0(\mathcal{H}, k) = 0$, $H_1(\mathcal{H}, k) = k[1 \otimes e_v] \oplus k[1 \otimes e_u]$, $H_2(\mathcal{H}, k) = k[1 \otimes e_v \wedge e_x] \oplus k[1 \otimes e_u \wedge e_x]$, and $H_n(\mathcal{H}, k) = 0$ for all $n \geq 3$. Moreover we find that the operator $B = (1 - \tau)\sigma N : H_1(\mathcal{H}, k) \longrightarrow H_2(\mathcal{H}, k)$ is bijective and we obtain

Theorem 5.1. *For any $q \in k$ which is not a root of unity one has*
$\widetilde{HC}_1^{(\delta,1)}(A(SL_q(2,k))) = k \oplus k$ *and* $\widetilde{HC}_n^{(\delta,1)}(A(SL_q(2,k))) = 0$ *for all $n \neq 1$.*
In particular, $\widetilde{HP}_0^{(\delta,1)}(A(SL_q(2,k))) = \widetilde{HP}_1^{(\delta,1)}(A(SL_q(2,k))) = 0$.

The above theorem shows that Theorem 4.1 is not true for non-cocommutative Hopf algebras.

The quantum universal enveloping algebra $U_q(sl(2,k))$ is an k-Hopf algebra which is generated as an k- algebra by symbols σ, σ^{-1}, x, y subject to the following relations

$$\sigma\sigma^{-1} = \sigma^{-1}\sigma = 1, \quad \sigma x = q^2 x\sigma, \quad \sigma y = q^{-2} y\sigma, \quad xy - yx = \frac{\sigma - \sigma^{-1}}{q - q^{-1}}.$$

The coproduct, counit and antipode of $U_q(sl(2,k))$ are defined by:

$$\Delta(x) = x \otimes \sigma + 1 \otimes x, \quad \Delta(y) = y \otimes 1 + \sigma^{-1} \otimes y, \quad \Delta(\sigma) = \sigma \otimes \sigma,$$

$$S(\sigma) = \sigma^{-1}, \quad S(x) = -x\sigma^{-1}, \quad S(y) = -\sigma y,$$

$$\epsilon(\sigma) = 1, \quad \epsilon(x) = \epsilon(y) = 0.$$

It is easy to check that $S^2(a) = \sigma a \sigma^{-1}$, so that (σ^{-1}, ϵ) is a modular pair in involution. As the first step to compute its cyclic homology we should find its Hochschild homology group with trivial coefficients. (k is a $U_q(sl(2,k))$ bimodule via ϵ). We define a free resolution for $\mathcal{H} = U_q(sl(2,k))$ as a $\mathcal{H}^e$-module as follows

$$(*) \qquad \mathcal{H} \xleftarrow{\;\mu\;} M_0 \xleftarrow{\;d_0\;} M_1 \xleftarrow{\;d_1\;} M_2 \xleftarrow{\;d_2\;} M_3 \ldots$$

Where M_0 is $\mathcal{H}^e$, M_1 is the free left $\mathcal{H}^e$-module generated by symbols $1 \otimes 1 \otimes e_\sigma, 1 \otimes 1 \otimes e_x, 1 \otimes 1 \otimes e_y$, M_2 is the free left $\mathcal{H}^e$-module generated by symbols $1 \otimes 1 \otimes e_x \wedge e_\sigma, 1 \otimes 1 \otimes e_y \wedge e_\sigma, 1 \otimes 1 \otimes e_x \wedge e_y$, and finally M_3 is generated by $1 \otimes 1 \otimes e_x \wedge e_y \wedge e_\sigma$ as a free left $\mathcal{H}^e$-module. We let $M_n = 0$ for all $n \geq 4$. We claim that with the following boundary operators, $(*)$ is a free resolution

for $\mathcal{H}$

$$d_0(1 \otimes 1 \otimes e_x) = x \otimes 1 - 1 \otimes x$$
$$d_0(1 \otimes 1 \otimes e_y) = y \otimes 1 - 1 \otimes y$$
$$d_0(1 \otimes 1 \otimes e_\sigma) = \sigma \otimes 1 - 1 \otimes \sigma$$
$$d_1(1 \otimes 1 \otimes e_x \wedge e_\sigma) = (\sigma \otimes 1 - 1 \otimes q^2\sigma) \otimes e_x - (q^2 x \otimes 1 - 1 \otimes x) \otimes e_\sigma$$
$$d_1(1 \otimes 1 \otimes e_y \wedge e_\sigma) = (\sigma \otimes 1 - 1 \otimes q^{-2}\sigma) \otimes e_y - (q^{-2}y \otimes 1 - 1 \otimes y) \otimes e_\sigma$$
$$d_1(1 \otimes 1 \otimes e_x \wedge e_y) = (y \otimes 1 - 1 \otimes y) \otimes e_x - (x \otimes 1 - 1 \otimes x) \otimes e_y$$
$$+ \frac{1}{q - q^{-1}}(\sigma^{-1} \otimes \sigma^{-1} + 1 \otimes 1) \otimes e_\sigma$$
$$d_2(1 \otimes 1 \otimes e_x \wedge e_y \wedge e_\sigma) = (y \otimes 1 - 1 \otimes q^2 y) \otimes e_x \wedge e_\sigma$$
$$- q^2(q^2 x \otimes 1 - 1 \otimes x) \otimes e_y \wedge e_\sigma + q^2(\sigma \otimes 1 - 1 \otimes \sigma) \otimes e_y \wedge e_x$$

To show that this complex is a resolution, we need a homotopy map. First we recall that the set $\{\sigma^l x^m y^n \mid l \in \mathbb{Z}, m, n \in \mathbb{N}_0\}$ is a P.B.W. type basis for $\mathcal{H}$ [7].
Let

$$\phi(a, b, n) = (a^{n-1} \otimes 1 + a^{n-1} \otimes b + \cdots + a \otimes b^{n-1} + 1 \otimes b^{n-1})$$

where $n \in \mathbb{N}, a \in \mathcal{H}, b \in \mathcal{H}^o$, and $\phi(a, b, 0) = 0$, and $\omega(p) = 1$ if $p \geq 0$ and 0 otherwise.

The following maps define a homotopy map for $(*)$ i.e. $Sd + dS = 1$:

$$S_{-1} : \mathcal{H} \to M_0,$$
$$S_{-1}(a) = 1 \otimes a,$$
$$S_0 : M_0 \to M_1,$$
$$S_0(\sigma^l x^m y^n \otimes b) = (1 \otimes b)((\sigma^l x^m \otimes 1)\phi(y, y, n) \otimes e_y +$$
$$+ (\sigma^l \otimes y^n)\phi(x, x, m) \otimes e_x) + \omega(l)(1 \otimes x^m y^n)\phi(\sigma, \sigma, l) \otimes e_\sigma$$
$$+ (\omega(l) - 1)(1 \otimes x^m y^n)\phi(\sigma^{-1}, \sigma^{-1}, -l)(\sigma^{-1} \otimes \sigma^{-1} \otimes e_\sigma),$$
$$S_1 : M_1 \to M_2,$$
$$S_1(\sigma^l x^m y^n \otimes b \otimes e_y) = 0,$$
$$S_1(\sigma^l x^m y^n \otimes b \otimes e_x) = (1 \otimes b)((\sigma^l x^m \otimes 1)\phi(y, y, n) \otimes e_x \wedge e_y$$
$$+ \frac{1 - q^{2n}}{(q - q^{-1})(1 - q^2)}(\sigma^l \otimes y^{n-1})\phi(x, x, m)(\sigma^{-1} \otimes \sigma^{-1} + q^{-2} \otimes 1) \otimes e_x \wedge e_\sigma$$
$$+ \frac{1}{q - q^{-1}}(\sigma^l x^m \otimes 1)\phi(y, y, n - 1)(\sigma^{-1} \otimes \sigma^{-1} + q^2 \otimes 1) \otimes e_y \wedge e_\sigma),$$
$$S_1(\sigma^l x^m y^n \otimes b \otimes e_\sigma) = (1 \otimes b)(q^2(\sigma^l x^m \otimes 1)\phi(y, q^2 y, n) \otimes e_y \wedge e_\sigma$$
$$+ q^{2(n-1)}(\sigma^l \otimes y^n)\phi(x, q^{-2} x, m) \otimes e_x \wedge e_\sigma),$$
$$S_2 : M_2 \to M_3,$$
$$S_2(a \otimes b \otimes e_x \wedge e_y) = 0,$$
$$S_2(a \otimes b \otimes e_y \wedge e_\sigma) = 0,$$
$$S_2(\sigma^l x^m y^n \otimes b \otimes e_x \wedge e_\sigma) = (1 \otimes b)(\sigma^l x^m \otimes 1)\phi(y, q^2 y, n) \otimes e_x \wedge e_y \wedge e_\sigma,$$
$$S_n = 0 : M_n \to M_{n+1} \text{ for } n \geq 3.$$

Again, by a rather long, but straightforward computation, we can check that $dS + Sd = 1$. By using the definition of Hochschild homology as $Tor^{\mathcal{H}^e}(\mathcal{H}, k)$ we have the following theorem

Theorem 5.2. $H_n(U_q(sl(2, k)), k) = k$ *if* $n = 0, 3$, *generated by* 1 *and* $1 \otimes e_x \wedge e_y \wedge e_\sigma$ *respectively, and* $H_n(U_q(sl(2, k)), k) = 0$ *for* $n \neq 0, 3$. *Here* k *is a* $U_q(sl(2, k))$-*bimodule via* ϵ *for both sides.*

Corollary 5.1. $\widetilde{HC}_n^{(\epsilon, \sigma)}(U_q(sl(2, k))) = k$ *when* $n \neq 1$, *and* 0 *for* $n = 1$.

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